

# Point-like Instantons and the $\text{Spin}(32)/\mathbb{Z}_2$ Heterotic String

Paul S. Aspinwall

Dept. of Physics and Astronomy,  
Rutgers University,  
Piscataway, NJ 08855

## Abstract

We consider heterotic string theories compactified on a K3 surface which lead to an unbroken perturbative gauge group of  $\text{Spin}(32)/\mathbb{Z}_2$ . All solutions obtained are combinations of two types of point-like instanton — one “simple type” as discovered by Witten and a new type associated to the “generalized second Stiefel-Whitney class” as introduced by Berkooz et al. The new type of instanton is associated to an enhancement of the gauge symmetry by  $\text{Sp}(4)$  and the addition of a massless tensor supermultiplet. It is shown that if four simple instantons coalesce at an orbifold point in the K3 surface then a massless tensor field appears which may be used to interpolate between the two types of instanton. By allowing various combinations of point-like instantons to coalesce, large gauge groups (e.g., rank 128) with many massless tensor supermultiplets result. The analysis is done in terms of F-theory.

# 1 Introduction

There has recently been considerable progress in the understanding of the nonperturbative physics of string compactification. A fairly realistic model which would be very nice to understand would be the heterotic string compactified on a Calabi–Yau threefold as this leads to an  $N = 1$  theory in four dimensions. Here we will deal with the more modest model of a heterotic string compactified on a K3 surface to yield an  $N = 1$  theory in six dimensions.

Starting with the work of [1, 2] it was realized that the structure of the heterotic string on a K3 surface could be related to the geometry of a Calabi–Yau threefold. In particular that the type IIA string on this Calabi–Yau space was dual to the heterotic string compactified on a product of a K3 surface and a 2-torus. Recall that the heterotic string requires a bundle structure for its compactification and so this product of K3 and a torus also comes equipped with a bundle.

By a process known as F-theory [3, 4, 5] one can analyze only the parts of the Calabi–Yau threefold,  $X$ , that are relevant to the K3 part of the compactification and ignore the 2-torus part. In order to do this,  $X$  must be in the form of an elliptic fibration with section  $p : X \rightarrow \Theta$ , where  $\Theta$  is a complex surface. For a precise statement of this see [6]. This may be viewed in two ways. Firstly one may take the area of the  $T^2$  to be large and switch off any Wilson lines around it and watch how  $X$  degenerates. Alternatively one may perform a fibre-wise mirror map and replace the type IIA theory with a type IIB string compactified on  $\Theta$ , where points in  $\Theta$  corresponding to “bad fibres” are identified with D-branes embedded in the space. Either way, F-theory promises to yield a fairly complete understanding of the entire moduli space of heterotic strings on a K3 surface.<sup>1</sup>

Since we are able to probe the moduli space so well, we should ask where the interesting points might be. An obvious place to look is where the underlying K3 surface itself degenerates to an orbifold. This, after all, has been where the interesting physics lives when one compactifies a type IIA or IIB string on a K3 surface [7, 8, 9]. It turns out that a simpler question to answer concerns when the bundle data on the K3 surface degenerates. This has no analogue for the type II string on the K3 surface.

The most obvious type of degeneration of a bundle is that of the “point-like instanton”. That is, where the curvature of the bundle is concentrated in an infinitesimal region of the base space. The study of such objects in heterotic string theory began with Witten’s paper [10]. Here it was argued that for the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string on a smooth K3 surface a point-like instanton induces massless vector multiplets enhancing the gauge symmetry by  $\mathfrak{sp}(1)$ .<sup>2</sup> This was suggested on general grounds, from the quaternionic nature of the moduli space of hypermultiplets and pictured in terms of Dirichlet 5-branes from the dual type I string theory. When  $k$  point-like instantons coalesce at the same point in the K3 surface it

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<sup>1</sup>Although the moduli associated with the R-R sector in the type IIA string may be troublesome.

<sup>2</sup>Unless we really need to, we will speak in terms of the algebra, rather than the group, of the gauge symmetry. This will allow us to ignore a lot of awkward  $\mathbb{Z}_2$  factors.

was argued that the gauge symmetry is enhanced by  $\mathfrak{sp}(k)$ .

In the context of point-like instantons, the  $E_8 \times E_8$  heterotic string appears, at first sight, to be a quite different animal. From duality to M-theory [11], it was argued in [12] that point-like instantons induce peculiar “tensionless strings” and new moduli in tensor supermultiplets which allow one to move off in a new direction in moduli space corresponding to a new massless tensor multiplet in the theory.

This is far from the end of the story for point-like instantons however. What if the underlying K3 surface is singular and a point-like instanton sits right at the singularity? This may well provide new behaviour. The purpose of this paper is to give an example of such an instanton in the context of the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string and explore some of its rich properties.

There are two approaches to nonperturbative analysis of the heterotic string by duality. One method, which we use here, is F-theory, which may be viewed as finding a type II dual. The rival method is that of using duality to the type I string, in particular by using orientifold methods [13]. One should note that one may directly relate these two approaches to each other of course [14]. The orientifold approach has proven very powerful in its ability to find explicit spectra for given models — see, for example, [15, 16]. Indeed, the subject of massless tensors associated to the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string was analyzed in this context in [17].

We wish to attempt to completely classify heterotic string theories on a K3 surface which contain the original ten-dimensional  $\text{Spin}(32)/\mathbb{Z}_2$  as part of their unbroken gauge symmetry. This will lead us to the new instanton. In general one should expect the F-theory approach to give a much better coverage of the moduli space of theories than the orientifold approach. This is because the orientifold approach necessarily focuses on points in the moduli space corresponding to the theories which are the global quotient of some other theory. One may also probe an infinitesimal region around this point by “twisted marginal operators”. F-theory on the other hand phrases questions in terms of elliptic fibrations. Since any smooth deformation of an elliptic threefold is also an elliptic threefold one might at first think one can probe the entire moduli space of a given theory. While this is almost true, current understanding of F-theory only treats enhanced gauge symmetry from the point of view of degenerate fibres. There is another potential contribution from the “Mordell-Weil group”. This arises when the fibration has an infinite number of sections. We will ignore this latter possibility.

It is not clear whether or not orientifold techniques could reproduce the results in this paper but it would be an interesting question to answer.<sup>3</sup>

We will present the classical geometry of this new instanton in section 2 and relate it to the “generalized second Stiefel-Whitney class” as introduced by Berkooz, Leigh, Polchinski, Schwarz, Seiberg, and Witten [18]. This will allow us to build our new “hidden obstructor”

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<sup>3</sup>I thank C. Johnson for a correspondence on this question.

point-like instanton in section 3 which has nonzero generalized second Stiefel-Whitney class but manages to not break any of the primordeal  $\text{Spin}(32)/\mathbb{Z}_2$  gauge group.

In section 4 we build the F-theory picture of the new instanton which allows us to determine its nonperturbative physics. The main result is the appearance of an  $\mathfrak{sp}(4)$  enhanced gauge symmetry and a massless tensor. In section 5 the F-theory picture is tied to known results about the  $E_8 \times E_8$  heterotic string and to the Gimon-Polchinski models.

In section 6 we show how to transform our new instantons into Witten's simple instantons and *vice versa*. This will also show that four simple instantons coalesced at an orbifold point in the K3 surface induce a massless tensor.

In section 7 we tackle the question of what happens when the two types of instantons hit each other. Large spectra of gauge symmetries and hypermultiplets appear. Finally we include an appendix which reviews some properties of elliptic threefolds that we require.

## 2 Bundles on K3 and the Kummer Lattice

Let us review the notion of a *generalized second Stiefel-Whitney class* following the work of Berkooz et al [18]. Consider a smooth  $G$ -bundle,  $E$ , on a smooth K3 surface,  $S$ . How can we express the topology of this bundle? Consider a 2-sphere,  $C$ , within  $S$  with a curve,  $\gamma$ , around its equator. An element,  $g_\gamma \in G$ , of the holonomy of  $E$  may be found by parallel transport around this curve.

View  $C$  as the union of its northern hemisphere,  $C_N$ , with its southern hemisphere,  $C_S$ . From the curvature,  $F$ , of  $E$  we may then determine

$$\begin{aligned} g_\gamma &= \exp \left( \int_{C_N} iF \right) \\ &= \exp \left( - \int_{C_S} iF \right). \end{aligned} \tag{1}$$

Thus

$$\int_C F = 2\pi n, \tag{2}$$

for some integer,  $n$ . Thus  $\frac{1}{2\pi}F$  appears as an element of  $H^2(S, \mathbb{Z})$ . This quantity will depend on the topology of  $E$  but it may be that different values of  $n$  specify the same topological class. To see how this works, consider the transition functions around  $\gamma$  from the northern hemisphere to the southern hemisphere as a map from  $\gamma$  into  $G$ . In order that  $E$  be homotopically nontrivial we require that the image of  $\gamma$  lie in a nontrivial element of  $\pi_1(G)$ . We may apply this construction to every homology 2-cycle within  $S$ .

We arrive at the result that a natural topological invariant of a  $G$ -bundle on  $S$  is given by a homomorphism from  $H_2(S, \mathbb{Z})$  to  $\pi_1(G)$ . If  $\pi_1(S)$  is trivial then the universal coefficients theorem [19] says that this group of homomorphisms is isomorphic to  $H^2(S, \pi_1(G))$ .

There are two very familiar examples of this invariant. First if  $E$  is the principle bundle of a holomorphic vector bundle then  $G \cong U(r)$ , for some  $r$ . Since  $\pi_1(U(r)) \cong \mathbb{Z}$  we have our invariant is simply an element of  $H^2(S, \mathbb{Z})$ . This is the *first Chern class*,  $c_1(E)$ . If  $E$  is the principle bundle of a real vector bundle then  $G \cong SO(r)$ . Since  $\pi_1(SO(r)) \cong \mathbb{Z}_2$  we have an object in  $H^2(S, \mathbb{Z}_2)$ . This is the *second Stiefel-Whitney class*,  $w_2(E)$ .

We are interested in the case  $G \cong \text{Spin}(32)/\mathbb{Z}_2$ . Clearly  $\pi_1(\text{Spin}(32)/\mathbb{Z}_2) \cong \mathbb{Z}_2$  and so we are in a situation analogous to the second Stiefel-Whitney class. Following [18] we denote this  $\tilde{w}_2 \in H^2(S, \mathbb{Z}_2)$  and consider it to be a “generalized second Stiefel-Whitney class”.

It will be convenient to represent  $\tilde{w}_2$  as a 2-cycle rather than a 2-cocycle. Dual to  $H^2(S, \mathbb{Z})$  is  $H_2(S, \mathbb{Z})$  in the usual way. We may then take  $H_2(S, \mathbb{Z})$  to be dual to itself by Poincaré duality. Thus we may take the dual of the dual of an element of  $H^2(S, \mathbb{Z}_2)$  as an element of  $H_2(S, \mathbb{Z}_2)$ . When seen this way,  $\tilde{w}_2$ , as an element of  $\text{Hom}(H_2(S), \mathbb{Z}_2)$ , may be viewed as

$$\tilde{w}_2 : C \rightarrow \#(\tilde{w}_2 \cap C) \pmod{2}, \quad (3)$$

where “ $\#$ ” represents the intersection number. We will simply use a dot to represent this natural inner product in  $H_2(S, \mathbb{Z})$  from now on.

We will be particularly interested in the case where the K3 surface,  $S$ , is a *Kummer Surface*. That is, when it has been obtained as the blow-up of the orbifold  $T^4/\mathbb{Z}_2$  in the usual way. The Kummer surface gives a natural set of elements in  $H_2(S, \mathbb{Z})$ . These are

1. The image of the six 2-cycles in the  $T^4$  under the quotient map.
2. The sixteen 2-spheres that appear as the exceptional divisors under blowing up.

Although these 22 elements may be used as a basis for  $H_2(S, \mathbb{Q})$ , they are not correctly normalized to form a basis for  $H_2(S, \mathbb{Z})$ . That is, they generate only a finite-index sublattice of  $H_2(S, \mathbb{Z})$ . This sublattice is called the *Kummer Lattice*.  $H_2(S, \mathbb{Z})$  is even self-dual, whereas the matrix of inner products on the generators of the Kummer Lattice has determinant not equal to one.

Let us use  $C_i, i = 1 \dots 16$ , to denote the sixteen exceptional divisors. Since  $C_i \cdot C_i = -2$  by the usual arguments (see, for example, [6]), then  $C_i/n$  cannot be an element of  $H_2(S, \mathbb{Z})$  for any integer,  $n > 1$ . Having said that, certain sums of  $C_i$ 's will be multiples of elements in  $H_2(S, \mathbb{Z})$ . This partially accounts for why  $\{C_i\}$  are not good generators for  $H_2(S, \mathbb{Z})$ . Let

$$D = \sum_{i=1}^{16} \xi_i C_i, \quad (4)$$

where  $\xi_i$  is either 0 or 1. One may then show that  $D$  will be *twice* an element of  $H^2(S, \mathbb{Z})$  if the following is true. The 16 exceptional divisors come from the 16 fixed points of the  $\mathbb{Z}_2$  action on  $T^4$ . The latter sixteen points naturally form the vertices of a hypercube. Consider

every two-dimensional face of the hypercube. Each such face will contain four  $C_i$ 's and thus can be associated with four  $\xi_i$ 's. The sum of these four  $\xi_i$ 's must be an even number.

A simple solution is to set all  $\xi_i$  to zero, which is trivial, or to set all  $\xi_i$  to one. The other possibilities correspond to having eight  $\xi_i$ 's equal to zero and eight  $\xi_i$ 's equal to one in suitable combinations.

In section 4.1 of [18] a picture of an instanton with  $\tilde{w}_2 \neq 0$  was given locally for an open neighbourhood of one of the exceptional divisors. This was given in terms of the curvature of the bundle which could be given compact support near the exceptional divisor. One may try to treat a  $\text{Spin}(32)/\mathbb{Z}_2$ -bundle as if it were a  $\text{Spin}(32)$ -bundle simply by viewing the transition functions as elements of  $\text{Spin}(32)$  rather than  $\text{Spin}(32)/\mathbb{Z}_2$ . As such we may try to build a bundle in the vector representation. Let the curvature of this resulting bundle be  $F_{\mathbf{32}}$ . One may then show that

$$\int_C \frac{1}{2\pi} F_{\mathbf{32}} = \frac{1}{2}(\tilde{w}_2 \cdot C) + n, \quad (5)$$

for some integer,  $n$ . Thus  $\tilde{w}_2$  can violate the quantization condition (2) and obstruct the existence of a vector representation — just as  $w_2$  obstructs a spin structure. If  $C_i$  is the exceptional divisor in question then the instanton of [18] satisfies

$$\int_{C_i} \frac{1}{2\pi} F_{\mathbf{32}} = \frac{1}{2}, \quad (6)$$

and thus obstructs a vector structure over  $C_i$ . We will call this instanton a “ $C_i$ -obstructor”. A  $C_i$ -obstructor can be seen to satisfy  $\tilde{w}_2 \cdot C_i = 1$ .

Since the curvature of a  $C_i$ -obstructor is meant to arise from local geometry,  $\tilde{w}_2$  should be proportional to  $C_i$ . Since  $C_i \cdot C_i = -2$ , we have

$$\tilde{w}_2 = \frac{1}{2} C_i. \quad (7)$$

Now let us fit the local  $C_i$ -obstructor picture into the global geometry of the K3 surface,  $S$ . It is clear that a single obstructor is not a valid configuration since  $\tilde{w}_2$  does not lie in integral homology. We may consider a situation where we place a single obstructor over more than one exceptional divisor. Now, if our set of exceptional divisors satisfies the Kummer lattice condition above then we are in business. The solution considered in [18] was to put an obstructor at all sixteen sites and so

$$\tilde{w}_2 = \frac{1}{2} \sum_{i=1}^{16} C_i, \quad (8)$$

which is in integral homology.

### 3 Unbroken Gauge Symmetry and Global Holonomy

When considering a compactification of a heterotic string on a bundle,  $E \rightarrow B$ , an important piece of information about  $E$  is its *global holonomy*,  $H$ . Let  $G_0$  be the “primordial” gauge group, i.e.,  $E_8 \times E_8$  or  $\text{Spin}(32)/\mathbb{Z}_2$ , of the heterotic string in ten dimensions. When compactified on  $E$ , this will be broken to the *centralizer* of  $H \subset G_0$ . That is, any element of  $G_0$  which commutes with all of  $H$  will remain a symmetry after compactification.

There are two contributions to the global holonomy group,  $H$ . Firstly there is the local holonomy generated by the curvature of  $E$ . Secondly there is the contribution from non-contractable loops from  $\pi_1(B)$  of the base space of  $E$ .

We will be most interested in the case where the global holonomy group is trivial and thus all of the primordial gauge symmetry remains in the lower-dimensional compactified theory. We need to make both the local holonomy and the contribution from  $\pi_1$  trivial.

We know from the work of [10] how to make the local holonomy trivial. Since this comes from the curvature of the bundle, we need to squeeze all of the region of the nonzero curvature into points over the base space. This limit is called “point-like instantons”. Of course, we haven’t really justified that the paths which happen to exactly pass through the point where a small instanton lives don’t pick up holonomy but the evidence is considerable [10, 20] that string theory really does allow one to ignore such paths.

Once we have shrunk all instantons down to zero size, we need only worry about non-contractable loops breaking the gauge group. Actually we should consider loops that are non-contractable *after* the points within  $B$  where the point-like instantons live have been removed since we are required to ignore paths which pass through such points.

Let us consider the case where  $B$  is a K3 surface. Since a K3 surface is simply connected, we need only worry about non-contractable loops produced by removing the locations of point-like instantons. If the instanton happens to sit at a smooth point inside the K3 surface then the open neighbourhood of the instanton, minus the point where it sits, may be retracted onto  $S^3$  — which is simply connected. Thus, a point-like instanton at a smooth point in a K3 surface breaks none of the primordial gauge group. For the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string, such point-like instantons are precisely the ones discovered by Witten [10]. We will denote such point-like instantons “simple”.

In the case that the K3 surface is a Kummer surface at an orbifold limit, we have a singularity locally of the form  $\mathbb{C}^2/\mathbb{Z}_2$ . If the instanton happens to be sat right on this singular point, then the neighbourhood retracts onto the lens space  $S^3/\mathbb{Z}_2$ . Since  $\pi_1(S^3/\mathbb{Z}_2)$  equals  $\mathbb{Z}_2$  we now have the possibility that the point-like instanton breaks part of the primordial gauge symmetry.

As shown in [18] this breaking of the gauge symmetry by  $\pi_1$  effects is intimately connected to  $\tilde{w}_2$  of the instanton. Let us review this fact. Consider blowing up the orbifold slightly so that we have an exception 2-sphere,  $C_i$ , in a small open neighbourhood of the K3 surface. We may also put the lens space  $S^3/\mathbb{Z}_2$  in this open neighbourhood, surrounding the 2-sphere.

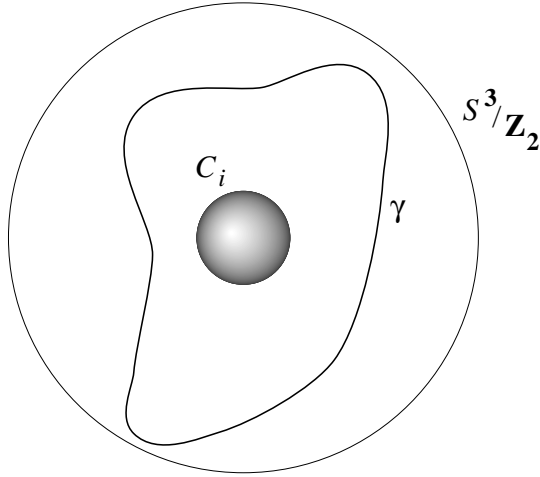


Figure 1: Lens space around an exceptional divisor.

We show this in figure 1.

Now, the lens space,  $L$ , may be viewed as an  $S^1$ -bundle over the 2-sphere,  $C_i$ . We may then use the Leray spectral sequence (see example 15.15 in [19]) to write the cohomology of  $L$  in terms of that of  $C_i$ . The important point is that there is an isomorphism

$$\phi : H^2(C_i, \mathbb{Z}_2) \cong H^2(L, \mathbb{Z}_2) \cong \mathbb{Z}_2. \quad (9)$$

This maps the topological class of bundles over  $C_i$  as measured by  $\tilde{w}_2$  into the class of bundles over  $L$  given by  $\phi(\tilde{w}_2)$ . The generator of  $H^2(L, \mathbb{Z}_2)$  may be associated to the generator of  $\pi_1(L)$ . This follows from the universal coefficients theorem [19] and the fact that  $H^2(L, \mathbb{Z})$  is pure torsion. Let us call this latter generator,  $\gamma$ . We show  $\gamma$  as a non-contractable loop in figure 1.

If  $\tilde{w}_2.C_i = 1$  then the bundle is nontrivial. The only way the bundle on  $L$  may be nontrivial is if the holonomy element generated by  $\gamma$  is nontrivial. Thus the global holonomy of the instanton is precisely measured by  $\tilde{w}_2$ .

As discussed in [18], the  $\mathbb{Z}_2$  subgroup of  $\text{Spin}(32)/\mathbb{Z}_2$  generated by  $\gamma$  is unique, up to endomorphisms. It is not the central  $\mathbb{Z}_2$  in  $\text{Spin}(32)/\mathbb{Z}_2$  and actually breaks the primordial gauge group down to  $\text{U}(16)/\mathbb{Z}_2$ . That is, *a point-like instanton in the form of a  $C_i$ -obstructor breaks  $\text{Spin}(32)/\mathbb{Z}_2$  to  $\text{U}(16)/\mathbb{Z}_2$ .*

This isn't what we want however. We want to see if we can leave the entire  $\text{Spin}(32)/\mathbb{Z}_2$  unbroken. There is a very simple way of producing a bundle with  $\tilde{w}_2 \neq 0$  and yet keeping the primordial  $\text{Spin}(32)/\mathbb{Z}_2$  intact. Consider the case where

$$\tilde{w}_2 = C_i. \quad (10)$$

Now we have  $\tilde{w}_2(C_i) = C_i.C_i \pmod{2} = 0$ . That is, the bundle over  $C_i$ , and hence  $L$ , is



now topologically trivial. The holonomy around  $\gamma$  will be trivial and so  $\text{Spin}(32)/\mathbb{Z}_2$  remains unbroken when  $C_i$  is blown down to a point. Let us call this the hidden- $C_i$ -obstructor.

At first sight it looks like we have constructed something rather trivial but consider the case where we have precisely one curve,  $C_i$ , in the K3 surface over which we put the hidden obstructor. Then as  $C_i$  is not twice an element of  $H^2(S, \mathbb{Z}_2)$  we really do have a nontrivial value for  $\tilde{w}_2$ . Following our discussion of the Kummer lattice in the previous section, the situation for hidden obstructers is somewhat the opposite as for the previously discussed non-hidden obstructers:

- Non-hidden obstructers must appear in multiples of eight so that the associated curves add up to twice an element of the Picard lattice.
- Hidden obstructers must *not* appear in such multiples of eight since they would then form a trivial bundle.

Let us recap the trick we have used here to find an instanton with nontrivial  $\tilde{w}_2$  which manages to keep the entire  $\text{Spin}(32)/\mathbb{Z}_2$  unbroken. Take, say, one exceptional  $S^2$ , call it  $C_i$ , and set  $\tilde{w}_2 = C_i$ . This value of  $\tilde{w}_2$  is trivial as far as  $C_i$  is concerned since the self-intersection of  $C_i$  is even. There will be another curve  $C_i^*$  dual to  $C_i$ , for which  $C_i^*.C_i = C_i^*.\tilde{w}_2 = 1$  and so  $\tilde{w}_2$  is nontrivial over this curve. Now go to the limit where we blow down  $C_i$ . We may put the support of the curvature near  $C_i$  as shown in [18] and so the curvature becomes zero everywhere except inside the point-like instanton. Thus, all that matters for global holonomy are the non-contractable loops around the lens space surrounding  $C_i$ . As we have shown, this is trivial. It is important to notice that  $C_i^*$  is *not* blown down during this process and so does not build a lens space which would pick up global holonomy.

In [18] it was shown that a single obstructing instanton locally contributes one to the second Chern class of the bundle. To build our hidden obstructor we essentially double the value of  $F_{32}$  in (6). Since  $c_2$  goes as the square of the curvature, we see that we multiply the second Chern class by four. That is, *the hidden obstructor contributes four to the second Chern class*.

We thus know the two possibilities for producing compactifications of the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string which preserve the  $\text{Spin}(32)/\mathbb{Z}_2$  gauge symmetry:

1. Point-like instantons at smooth points in the K3 surface which have instanton number (i.e., contribution to  $c_2$ ) one and  $\tilde{w}_2 = 0$ .
2. Point-like instantons stuck at orbifold points in the K3 surface which have instanton number four and  $\tilde{w}_2 \neq 0$ .

## 4 The Dual Picture

To understand nonperturbatively how string theory behaves on our new instantons we need to find a dual picture. This is provided by F-theory. Recall that F-theory associates a

Calabi–Yau threefold,  $X$ , to a heterotic string on a K3 surface [3, 4]. For F-theory to work,  $X$  must be in the form of an elliptic fibration  $p : X \rightarrow \Theta$ . One may regard the heterotic string on the K3 surface as dual to either the type IIB string on  $\Theta$  with some D-brane insertions or, alternatively, to some special large radius limit of the type IIA string on  $X$ . Either way, the non-perturbative physics of the heterotic string becomes encoded in the elliptic fibration,  $p : X \rightarrow \Theta$ . See [5, 6] for more details. In particular, we will use the notation from [6] and assume a knowledge of many of the results in section 6 of that paper.

We would like to completely classify all heterotic string compactifications on a K3 surface which lead to a gauge symmetry containing  $\text{Spin}(32)/\mathbb{Z}_2$  in the perturbatively-understood part of the gauge symmetry. The only assumption we will make (subject to a few caveats outlined in [6]) is that none of this gauge symmetry arises from the Mordell-Weil group.

We wish to understand theories which at least begin as a perturbatively-understood heterotic string theory. We thus want to begin with one tensor multiplet and, as such, we assume  $\Theta$  is of the form of a Hirzebruch surface  $\mathbf{F}_n$ . The Hirzebruch surface is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}_1$  with a natural zero section,  $C_0$ . We will denote the class of the fibre,  $f$ . We will call such fibres, “ $f$ -curves”, to avoid any confusion with the fibres of  $X$  as an elliptic fibration. To obtain an  $\mathfrak{so}(32)$  term in the gauge algebra we require a line of  $I_{12}^*$  fibres in  $\Theta$ . To make this  $\mathfrak{so}(32)$  part of the perturbatively-understood symmetry we put it along a section of  $\mathbf{F}_n$ . Let us assume it is the zero section,  $C_0$ . We may do this without loss of generality so long as we do *not* impose  $n \geq 0$ .

To make the group precisely  $\text{Spin}(32)/\mathbb{Z}_2$ , it was shown in [20] that one required  $X$  to have precisely two global sections, as an elliptic fibration. This forces a factorization of the Weierstrass form of the elliptic fibration:

$$\begin{aligned} y^2 &= x^3 + ax + b \\ a &= q - p^2 \\ b &= -pq \\ \delta &= (q + 2p^2)^2(4q - p^2), \end{aligned} \tag{11}$$

where  $p$  and  $q$  are functions over  $\mathbf{F}_n$  and  $\delta$  is the discriminant. The  $I_{12}^*$  condition forces  $(a, b, \delta)$  to vanish to order  $(2, 3, 18)$  along  $C_0$ . Denote

$$\begin{aligned} m_1 &= q + 2p^2 \\ m_2 &= 4q - p^2. \end{aligned} \tag{12}$$

One may then show that  $m_1$  must vanish to order 8 along  $C_0$  and  $m_2$  vanishes to order 2. Let us use upper case letters to denote the divisors in  $\mathbf{F}_n$  associated to the various functions above. The Calabi–Yau condition imposes

$$\begin{aligned} \Delta &= 2M_1 + M_2 \\ M_1 &= M_2 = 8C_0 + (8 + 4n)f. \end{aligned} \tag{13}$$

Let us split off from  $M_1$  and  $M_2$  the parts giving the  $I_{12}^*$  along  $C_0$ :

$$\begin{aligned} M_1 &= M'_1 + 8C_0 \\ M_2 &= M'_2 + 2C_0, \end{aligned} \tag{14}$$

with  $\Delta'$  defined similarly. Now, to make sure that the fibres along  $C_0$  are generically nothing worse than  $I_{12}^*$ , neither  $M'_1$  nor  $M'_2$  should contain any more of  $C_0$ . This means that the intersection numbers

$$\begin{aligned} M'_1.C_0 &= 8 + 4n \\ M'_2.C_0 &= 2(4 - n), \end{aligned} \tag{15}$$

must be nonnegative. Thus  $-2 \leq n \leq 4$ .

Let us treat the remainder of the discriminant, given by  $M'_1$  and  $M'_2$  in turn.  $M'_1$  is simply  $8 + 4n$  copies of  $f$ . Generically this will mean it is  $8 + 4n$  parallel lines along the  $f$  direction. As  $\Delta$  contains  $2M'_1$ , this will produce lines of  $I_2$  fibres. Thus, the gauge symmetry is enhanced nonperturbatively by  $8 + 4n$   $\mathfrak{sp}(1)$  terms. These are precisely Witten's simple point-like instantons of [10].

If  $k$  of these instantons are brought together,  $k$  lines of  $I_2$  will merge to form a line of  $I_{2k}$ . As explained in [20], monodromy turns the  $\mathfrak{su}(2k)$  gauge algebra one might first associate to this into an  $\mathfrak{sp}(k)$  gauge algebra. One can potentially have monodromy whenever a curve in the discriminant, whose associated gauge algebra may admit nontrivial outer automorphisms, collides with another component of the discriminant. Whether or not there is monodromy can be determined purely in terms of the local geometry of the collision, and with what type of curve it collided. In our case we have a transverse collision of a line of  $I_{12}^*$  fibres with a line of  $I_{2k}$  fibres. One may show that such a collision induces no monodromy in the  $I_{12}^*$  fibre but has a  $\mathbb{Z}_2$  action in the  $I_{2k}$  fibre. We show how to determine how the monodromy acts in the appendix.

As well as the gauge algebra, we may also determine the spectrum of hypermultiplets as discussed in [20, 21, 22]. The transverse collision of the  $I_{12}^*$  and  $I_{2k}$  produce a half hypermultiplet in the  $(\mathbf{32}, \mathbf{2k})$  representation of  $\mathfrak{so}(32) \oplus \mathfrak{sp}(k)$ .

As discussed in [10] we should also expect a hypermultiplet in the  $\mathbf{k}(2\mathbf{k} - \mathbf{1}) - \mathbf{1}$  (i.e., antisymmetric tensor) representation of  $\mathfrak{sp}(k)$ . Call this the  $\mathbf{A}_2$  representation for brevity. Let us use  $\Delta''$  to denote the discriminant after the contribution from  $C_0$  and all the  $f$ -curves has been subtracted. To see how the hypermultiplets arise note that an  $f$ -curve is topologically a sphere. Thus, if we are to have a nontrivial action of monodromy on the fibre of the elliptic fibration around this sphere, we must have more than one branch point. At present we have only found one collision — that of  $f$  with  $C_0$ . There must be further collisions of  $\Delta''$  with the  $I_{2k}$  line to produce more monodromy. As explained by Morrison [23], these collisions will produce the  $\mathbf{A}_2$  representation required.

To see this we use the results of [24] which say that if a curve of bad fibres is of genus  $g$ , then we expect  $g$  hypermultiplets in the adjoint of the associated gauge algebra, in addition

to the usual adjoint of vectors. When monodromy acts within the curve, the algebra is split between the part invariant under the monodromy and the rest which varies. The vectors are only associated with the monodromy-invariant part (see, for example, [6]) but we may pick up hypermultiplets in the part that varies depending on the genus of the base curve after we have taken the monodromy into account. Thus, suppose we have a  $\mathbb{Z}_2$  monodromy acting on a rational curve in  $\Theta$  associated to a gauge algebra (before monodromy is taken into account)  $\mathfrak{g}$ . The outer automorphism induced by the monodromy leaves  $\mathfrak{g}_0$  invariant. The adjoint of  $\mathfrak{g}$  may then be decomposed into the adjoint of  $\mathfrak{g}_0$  plus a representation  $R'$ . Suppose the monodromy is branched over  $n_p$  points within the rational curve. Then as far as the representation  $R'$  is concerned, the base curve is actually a double cover of the rational curve branched at  $n_p$  points. This has genus  $\frac{1}{2}n_p - 1$ . We therefore expect  $\frac{1}{2}n_p - 1$  hypermultiplets in the  $R'$  representation.

In our case, we are reducing  $\mathfrak{su}(2k)$  to  $\mathfrak{sp}(k)$ . It is easy to show that  $R'$  is indeed the  $\mathbf{A}_2$  representation. Now we need to know how many points there are within each  $f$ -curve over which the  $\mathbb{Z}_2$  monodromy is branched. This will allow us to count the  $\mathbf{A}_2$ 's.

Let us introduce affine coordinates  $(s, t)$  to parameterize  $\mathbf{F}_n$  locally. Let  $C_0$  be given by  $s = 0$  and let us fix a particular  $f$ -curve to be given by  $t = 0$ . To associate an  $\mathfrak{sp}(k)$  gauge symmetry with this  $f$ -curve we require  $m_1$  to be of order  $k$  in  $t$ . Let us put  $m_1 = t^k$  for the simplest case. Then

$$\delta = t^{2k}(4t^k - 9p^2). \quad (16)$$

Thus,  $\Delta$  will collide with this  $f$ -curve whenever  $p(s, t)$  has a zero.  $P$  is in the class  $4C_0 + (4 + 2n)f$  and thus collides with  $f$  a total of  $P \cdot f = 4$  times. One of these collisions is the transverse collision with the line of  $I_{12}^*$  fibres along  $C_0$ . The other three collision are generically non-transverse collisions with a curve of  $I_1$  fibres along  $\Delta''$ . As we will see in the appendix, all four collisions induce monodromy — we have  $n_p = 4$  and thus one hypermultiplet in the  $\mathbf{A}_2$  representation as desired.

Thus far we have recovered the simple  $\mathfrak{sp}(1)$  point-like instantons. Now we discover something new when we look at collisions between  $\Delta''$  and  $C_0$ . Since  $M_1$  is order 8 along  $C_0$  as explained above, let us put  $m_1 = s^8$ . It follows that

$$\delta = s^{16}(4s^8 - 9p^2). \quad (17)$$

We know that  $m_2$  vanishes to order 2 along  $s = 0$  so we must be able to factorize  $p = sp_1$ . Therefore

$$\delta = s^{18}(4s^6 - 9p_1^2). \quad (18)$$

Thus there will be collisions between  $\Delta''$  and  $C_0$  whenever  $p_1$  has extra zeros. This happens at  $C_0 \cdot P_1 = 4 - n$  points.

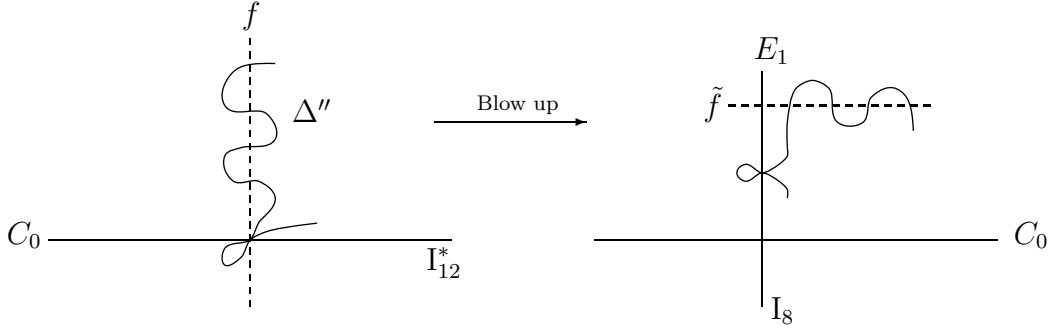


Figure 2: Blowing up the  $\Delta''$  collision with  $C_0$ .

Put  $p_1 = t$  to get a local form of the collision. (Note that now  $t = 0$  is *not* the equation of an  $f$ -curve for simple instantons.) Adding the degrees in  $s$  and  $t$  together, we see that  $(a, b, \delta)$  have degrees  $(4, 6, 20)$  respectively. As explained in [6], whenever these degrees are greater than, or equal to,  $(4, 6, 12)$ , one must blow-up the base to resolve  $X$ . Therefore, these collision of  $\Delta''$  with  $C_0$  induce new massless *tensor* degrees of freedom.

Let  $E_1$  be the resulting exceptional  $\mathbb{P}^1$  in the blown-up  $\Theta$ . The order to which  $(a, b, \delta)$  vanish over generic points in  $E_1$  is given by subtracting  $(4, 6, 12)$  from the orders at the point which was blown-up. That is, the orders are  $(0, 0, 8)$ . Thus we have  $I_8$  fibres along  $E_1$ . The collision between  $E_1$  and  $C_0$  produces monodromy. This results in a gauge symmetry of  $\mathfrak{sp}(4)$ .

As shown in figure 2,  $\Delta''$  collides with  $E_1$  just once to produce another monodromy branch point. This means that  $n_p = 2$  for this  $\mathfrak{sp}(4)$  gauge symmetry and so no hypermultiplets in the  $\mathbf{A}_2$  representation appear. There will be hypermultiplets in the  $(\mathbf{32}, \mathbf{8})$  representation from the collision of  $E_1$  with  $C_0$ .

Let us review the spectrum we have obtained. Let the simple instantons clump together in  $\mu$  groups of  $k_i$  (so that  $\sum_{i=1}^{\mu} k_i = 8 + 4n$ ) but leave the collisions of  $C_0$  and  $\Delta''$  isolated. We have

1. A gauge algebra

$$\mathfrak{so}(32) \oplus \left( \bigoplus_{i=1}^{\mu} \mathfrak{sp}(k_i) \right) \oplus \mathfrak{sp}(4)^{\oplus(4-n)}. \quad (19)$$

2.  $5 - n$  massless tensor supermultiplets (including the dilaton).
3. Hypermultiplets (or half-hypermultiplets if the representation is not complex) in the

following representations:

$$\begin{array}{lll}
(\mathbf{32}, 2\mathbf{k}_i) & \text{of} & \mathfrak{so}(32) \oplus \mathfrak{sp}(k_i) \\
\mathbf{k}_i(2\mathbf{k}_i - \mathbf{1}) - \mathbf{1} & \text{of} & \mathfrak{sp}(k_i) \\
(\mathbf{32}, \mathbf{8}) & \text{of} & \mathfrak{so}(32) \oplus \mathfrak{sp}(4) \quad (4 - n \text{ times})
\end{array} \tag{20}$$

as well as some chargeless hypermultiplets.

At this point the interpretation of this model in terms of point-like instantons discussed at the end of section 3 should be fairly evident. As mentioned above, the simple instantons are associated to the  $8 + 4n$  zeros of  $m_1$ . Each of these have instanton number one. If we assign instanton number four to each of the  $4 - n$  collisions between  $C_0$  and  $\Delta''$  then

$$\sum_{i=1}^{\mu} k_i + 4(4 - n) = 24, \tag{21}$$

for the total instanton number as expected for the bundle,  $E$ , on a K3 surface. *We therefore identify these  $4 - n$  collisions as point-like hidden-obstructer instantons in the dual heterotic string.*

We see that each hidden obstructer instanton is associated to a massless tensor multiplet and an  $\mathfrak{sp}(4)$  gauge symmetry.

Let us be precise about what we mean exactly by this statement. As discussed in [12], phase transitions between tensor moduli and hypermultiplet moduli are fairly exotic in nature. This is exactly what we have here — when the size of the new instanton is shrunk down to zero size (by hypermultiplets) a new modulus appears as the scalar component of a tensor supermultiplet which allows us to move off into a new component of the moduli space. Rather than speak of the theory right at the phase transition point, which has “tensionless strings” roughly speaking, we will assume that we switch on the new tensor modulus slightly to move away from this peculiar theory. As a result, we have a more conventional six-dimensional theory (although it has no covariant action) and we may ask sensible questions about anomalies etc. We shall not attempt to say anything in this paper about the theory which sits right on the phase transition point.

To complete the spectrum we should count the number of chargeless hypermultiplets. Roughly speaking, this is given by the number of deformations of complex structure of  $X$  plus one. One needs to be a little careful however. It may be that F-theory counts some of the linear combinations of charged hypermultiplets which can also act as deformations. This latter effect is due to the appearance of “elliptic scrolls” in  $X$  and is tied to Wilson’s work on the phenomenon of the Kähler cone jumping for special values of complex structure [25] (see also [6] for a brief account of this). This indeed happens when simple instantons coalesce. To avoid this issue let us assume all  $8 + 4n$  simple instantons are isolated.

We know  $h^{1,1}(X)$  from the blow-ups in both the base (the tensor multiplets) and the fibre (the rank of the gauge group). We have  $h^{1,1} = 3 + (4 - n) + 24 + 16 = 47 - n$ . To calculate

$h^{2,1}(X)$  we need the Euler characteristic of  $X$ . This is done by adding the contributions from all of the bad fibres in  $X$  as an elliptic fibration. For an example see [6]. In our case we need some Euler characteristics of some of the fibres appearing over collisions within  $\Delta$ . We calculate those required in the appendix. The result is<sup>4</sup>

$$\begin{aligned}\chi(X) &= \sum_{i=1}^{\mu} \{2k_i.(2-4)\} + 18.(2-\mu-4+n) + 8.(4-n).(2-2) \\ &\quad + (-24-3n-3\mu-3\mu-(4-n)) + 3 \sum_{i=1}^{\mu} (2+k_i) + \sum_{i=1}^{\mu} (18+k_i) \quad (22) \\ &\quad + (4-n).22 + (4-n).6 \\ &= 48 - 12n.\end{aligned}$$

This gives  $h^{2,1} = h^{1,1}(X) - \frac{1}{2}\chi(X) = 23 + 5n$ . Therefore there are  $24 + 5n$  chargeless hypermultiplets.

As always, one may check this F-theory calculation to ensure that anomalies cancel (as they must). The gravitational anomaly yields

$$\begin{aligned}273 - 29n_T - n_H + n_V &= 273 - 29.(5-n) - \frac{1}{2}.2.32.24 - (24+5n) \\ &\quad + 3.(8+4n) + 36.(4-n) + 496 \quad (23) \\ &= 0.\end{aligned}$$

Similarly one may check the gauge anomalies.<sup>5</sup>

This counting of chargeless hypermultiplets fits nicely with the heterotic interpretation. The underlying K3 surface has 20 (quaternionic) deformations and the simple instantons may be placed anywhere giving  $8 + 4n$  more deformations. Each hidden obstruder requires an orbifold point locally of the form  $\mathbb{C}^2/\mathbb{Z}_2$ , which reduces the number of deformations of the K3 by one. The location of each hidden obstruder is then fixed at this orbifold point. Thus, the total number of deformations is  $20 + (8 + 4n) - (4 - n) = 24 + 5n$  as expected.

One may also check the above calculations in the case that some of the simple instantons coalesce. In this case the topology of  $X$  is actually unchanged but the interpretation of some of the hypermultiplets is modified.

## 5 Some Equivalences

Recall the behaviour of the  $E_8 \times E_8$  heterotic string on a K3 surface as regards F-theory [12, 4, 5]. The topology of the required  $E_8 \times E_8$ -bundle on the K3 surface is specified by

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<sup>4</sup>The only awkward step is calculating the Euler characteristic of  $\Delta''$ . This is done by using the adjunction formula and then compensating for the various high-order tacnodes which appear in  $\Delta''$ .

<sup>5</sup>I thank N. Seiberg for explaining this to me and showing that they do indeed cancel.

how the total second Chern class is split between the two  $E_8$ 's. In particular, F-theory on the Hirzebruch surface  $\mathbf{F}_n$  is dual to a split of  $12 + n$  and  $12 - n$ .

This shows the T-duality between the  $E_8 \times E_8$  heterotic string on a K3 surface and the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string on another K3 surface. For example, as has been known for some time [4], the  $\text{Spin}(32)/\mathbb{Z}_2$  heterotic string with  $\tilde{w}_2 = 0$  must be dual to the  $E_8 \times E_8$  string with the second Chern class split  $(8, 16)$  between the two  $E_8$ 's. This follows since  $\tilde{w}_2 = 0$  implies that there can be no obstructers, hidden or non-hidden, which implies that  $4 - n = 0$ .

This raises a point which, at least at first sight, looks puzzling. The global diffeomorphisms of the underlying K3 surface, on which the heterotic string lives, can transform one value of  $\tilde{w}_2$  into another. In particular there are only three equivalence classes once this is taken into account [18]:

1.  $\tilde{w}_2 = 0$ ,
2.  $\tilde{w}_2 \neq 0$  and  $\tilde{w}_2 \cdot \tilde{w}_2 = 0 \pmod{4}$ ,
3.  $\tilde{w}_2 \neq 0$  and  $\tilde{w}_2 \cdot \tilde{w}_2 = 2 \pmod{4}$ .

Since  $(4 - n)$  hidden obstructers over disjoint  $(-2)$ -curves yields  $\tilde{w}_2 \cdot \tilde{w}_2 = 2(n - 4)$  we are implying equivalences between certain  $E_8 \times E_8$  string vacua. Actually these equivalences do exist.

To see this we need to look at the strange properties of the Hirzebruch surface.<sup>6</sup> The topology of  $\mathbf{F}_n$  is actually only specified by whether  $n$  is even or odd. Indeed one may build a family of surfaces  $\pi : Z \rightarrow D$ , where  $D$  is a complex disc with coordinate  $z$  such that the fibre at  $z \neq 0$  is  $\mathbf{F}_n$  but at  $z = 0$  it becomes  $\mathbf{F}_{n+2}$ . At  $z = 0$  a new algebraic curve within the fibre jumps into existence with self-intersection  $-n - 2$ . This causes the Kähler cone to contract, relative to that of  $\mathbf{F}_n$ , but nothing has changed topologically.

This equivalence between Hirzebruch surfaces is used to show the equivalence of the  $n = 0$  model and the  $n = 2$  model as in [26, 4]. The elliptic threefold fibred over  $\mathbf{F}_2$  is a codimension one subset (which can be realized as a hypersurface in a weighted projective space) of the more general member of the family which is fibred over  $\mathbf{F}_0$ . The jumping Kähler cone of the Hirzebruch surface is transferred to the threefold whose Kähler cone also shrinks over this special sub-family.

It was shown in [25] that jumping Kähler cones could only happen in smooth threefolds if the algebraic class that jumped into existence for special values of the complex structure was an “elliptic scroll”. That is, an elliptic curve times a rational curve. Thus, this rational curve is a  $(0, -2)$ -curve within the Calabi–Yau threefold. Within the base of an elliptic fibration therefore, the only curve which is allowed to jump into existence is a  $(-2)$ -curve,

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<sup>6</sup>I thank D. Morrison for conversations about this.



which would come from the Hirzebruch surface  $\mathbf{F}_2$ . We appear to have shown that the only equivalence allowed between models is the  $n = 0$  to  $n = 2$  equivalence.

We may obtain the rest of the equivalences by relaxing the constraint that the elliptic threefold be smooth. Now any smooth transition between  $\mathbf{F}_n$  and  $\mathbf{F}_{n+2}$  may be turned into a “smooth” transition between singular elliptic threefolds. In the case we are studying the Calabi–Yau threefold has a curve of  $D_{16}$ -type singularities inducing the  $\text{Spin}(32)/\mathbb{Z}_2$  gauge symmetry. It is certainly singular.

Given this equivalence between Calabi–Yau threefolds, there is no contradiction between  $\tilde{w}_2$  equivalence classes and F-theory equivalence classes.

Now let us turn our attention to the connection between the hidden obstructer theories we have described in terms of F-theory and other models in the same  $\tilde{w}_2$  class which break at least part of  $\text{Spin}(32)/\mathbb{Z}_2$ . We focus on the Gimon-Polchinski models of [13]. As explained in [18], we expect these models to all be in the F-theory class with  $n = 0$ .

To see this simply deform  $m_2$  of (12) so that it no longer vanishes along  $C_0$ . This will turn the line of  $\mathbf{I}_{12}^*$  line of fibres along  $C_0$  into a line of  $\mathbf{I}_{16}$  fibres. This changes the class of  $\Delta''$  but it will still collide with  $C_0$  at four points (doubly at each point). These collisions will induce monodromy and so the  $C_0$  line now generates a  $\mathfrak{sp}(8)$  gauge symmetry.

This breaking of  $\text{Spin}(32)/\mathbb{Z}_2$  may be seen by the maximal subgroup

$$\frac{\text{Spin}(32)}{\mathbb{Z}_2} \supset \text{SO}(3) \times \frac{\text{Sp}(8)}{\mathbb{Z}_2} \times \mathbb{Z}_2. \quad (24)$$

Giving the hidden obstructer nonzero size can turn it into a smooth  $\text{SO}(3)$ -bundle. (The group must be non-simply-connected since  $\tilde{w}_2 \neq 0$ .) Thus the global holonomy breaks the primordial gauge symmetry to  $\text{Sp}(8)/\mathbb{Z}_2$  consistent with what we saw from F-theory.

Further deformations can be used to bunch the four points of collision between  $\Delta''$  and  $C_0$  into two coalesced pairs. This will remove the monodromy and so the gauge symmetry  $\mathfrak{sp}(8)$  will turn into  $\mathfrak{su}(16)$ .<sup>7</sup> To fit in with the work of [13] (see also the earlier work of [28]) we may then identify the two points of collision of  $\Delta''$  with  $C_0$  as each yielding a hypermultiplet in the **120** of  $\mathfrak{su}(16)$ .

The line of fibres along  $C_0$  can be broken up into a parallel set of lines of  $\mathbf{I}_{2l_j}$  fibres so that  $\sum_j l_j = 8$ . This makes the class  $C_0$  analogous to the class  $f$  in which we have a set of parallel lines of  $\mathbf{I}_{2k_i}$  fibres satisfying  $\sum_i k_i = 8$ . This is as it should be since  $\mathbf{F}_0$  has an obvious symmetry between the classes  $C_0$  and  $f$ .

This allows us to reproduce all of the Gimon-Polchinski models in terms of F-theory. We show an example in figure 3. Note that the collisions within the discriminant will produce massless hypermultiplets in various representations in the usual way.

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<sup>7</sup>Actually there are good reasons to expect the Mordell-Weil group to enhance this further to  $\mathfrak{u}(16)$ . This is because  $X$  can be written as a K3 fibration whose generic fibre can be written as a double cover of a rational elliptic surface. The rational elliptic surface thus obtained is known from the classification of [27] to have a Mordell-Weil group of rank one. I thank M. Gross for conversations on this point.

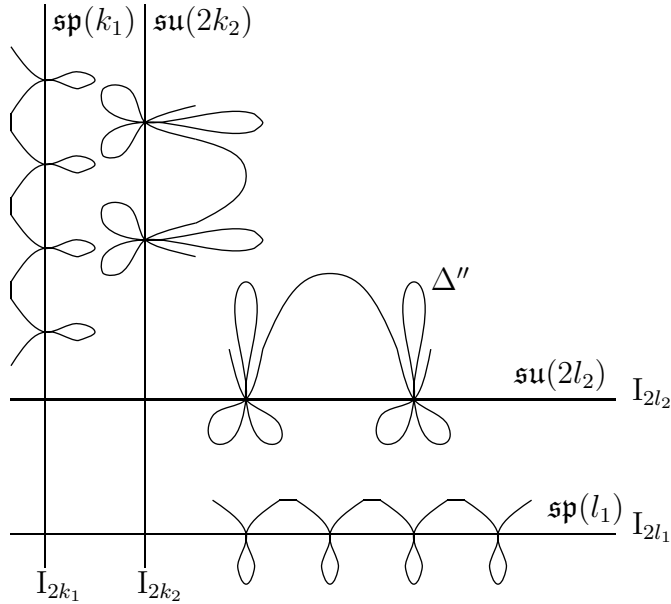


Figure 3: F-theory picture of Gimon-Polchinski models.

## 6 Phase Transitions

Let us return to the  $E_8 \times E_8$  heterotic string with its second Chern class split as  $12 + n$  and  $12 - n$  between the two  $E_8$ 's. When any of the instantons become point-like in the  $E_8 \times E_8$  a new massless tensor supermultiplet appears. One may then use this new direction in the moduli space to move to another theory with a point-like instanton with the second Chern class split  $(11 + n, 13 - n)$ . This instanton can then be given size to remove the massless tensor. Thus, by a process that involves two phase transitions, we may effectively change the topology of the  $E_8 \times E_8$ -bundle.

In terms of M-theory [12] this was understood by a 5-brane peeling itself off one “end of the universe” and moving over (by varying the tensor degree of freedom) to the other end of the universe.

In terms of F-theory [5], which is the approach we use here, this is achieved by first blowing up a point in  $\mathbf{F}_n$ . The proper transform of the fibre that passed through this point then has self-intersection  $-1$  allowing it to be blown down. This blow down results in the Hirzebruch surface  $\mathbf{F}_{n\pm 1}$  (depending on whether the original point blown up was on  $C_0$  or not).

Our new point-like hidden obstructer instanton is very similar to the point-like  $E_8$  instanton in that a new massless tensor results. We may therefore follow a phase transition to another Hirzebruch surface and see what happens. We will discover that we may transform hidden obstructer instantons into simple instantons and *vice versa*.

Begin with the collision of  $\Delta''$  with  $C_0$  in the Hirzebruch surface,  $\mathbf{F}_n$ , as in the previous section. As we discussed above, to resolve  $X$ , such a collision must be blown-up within  $\mathbf{F}_n$ . The exceptional divisor,  $E_1$ , results. Let  $\tilde{f}$  be the proper transform of the  $f$ -curve that passed through the collision. As blow-ups decrease self-intersections by one and  $f.f = 0$ , we see that  $\tilde{f}.\tilde{f} = -1$ .

We know that  $\Delta''.f = 6$  and that the collision with  $C_0$  accounts for two of these intersections. Thus, away from  $C_0$ ,  $\Delta''$  hits our particular  $f$ -curve four times. Assuming everything else is generic, these will be at four distinct points. Thus the proper transform of  $\Delta''$ , which we also denote  $\Delta''$ , hits  $\tilde{f}$  at four distinct points.

We may now blow down  $\tilde{f}$ . This gives the proper transform of  $E_1$  a self-intersection of 0 and it becomes a fibre,  $f$ , of the Hirzebruch surface  $\mathbf{F}_{n+1}$  which we have now made. Now  $\Delta''$  will hit this new  $f$ -curve four times at the same point (where  $\tilde{f}$  used to hit  $E_1$ ).

Now deform  $X$  so that this quadruple collision of  $\Delta''$  with  $f$  divides into two double collisions. What we have done is to produce exactly the F-theory picture of four coalesced simple instantons giving a gauge group  $\mathfrak{sp}(4)$ .

Let us repeat what we have done in the language of the heterotic string. Begin with a point-like hidden obstructer instanton. Then move along in moduli space from one phase to another using the massless tensor degree of freedom. Then deform using hypermultiplets to get rid of the massless tensor. The hidden obstructer has disappeared ( $n$  has increased by one) but four new simple instantons have appeared.

We see therefore that our two types of point-like instantons may be transformed into each other by using massless tensors. This also gives a way of changing the topology (as given by  $\tilde{w}_2$ ) of the associated vector bundle. Thus we see that the picture is very analogous to the  $E_8 \times E_8$  heterotic string.

Note that the geometry of the K3 surface is given by hypermultiplet deformations and so is fixed while we vary the tensor. As we knew that the hidden obstructer lived on an orbifold point, the orbifold point must still be there after moving along the tensor direction. What's more we know that the location of the simple instantons are also given by hypermultiplets. This means that, before we get rid of the massless tensor by moving the simple instantons, the four simple instantons must have been sat right on the orbifold point. This implies that *four simple instantons on a  $\mathbb{C}^2/\mathbb{Z}_2$  quotient singularity in the K3 surface produce a massless tensor supermultiplet.*

In figure 4 we show the phase transition we described above backwards. Start with a heterotic string with, say  $\tilde{w}_2 = 0$  (and therefore  $\Theta \cong \mathbf{F}_4$ ). Then bring four simple point-like instantons together to form a theory containing a gauge symmetry of  $\mathfrak{sp}(4)$ . Now give the K3 surface a  $\mathbb{C}^2/\mathbb{Z}_2$  quotient singularity and put this coalesced instanton at that point. Now a massless tensor appears. Use this massless tensor to turn  $\mathbf{F}_4$  into  $\mathbf{F}_3$ . Now we have a hidden obstructer instanton at the orbifold point. If we wish, the massless tensor may be given mass by giving the new point-like instanton size (which will break  $\mathfrak{so}(32)$ ).

A natural question to ask is what happens if fewer than four simple instantons coalesce

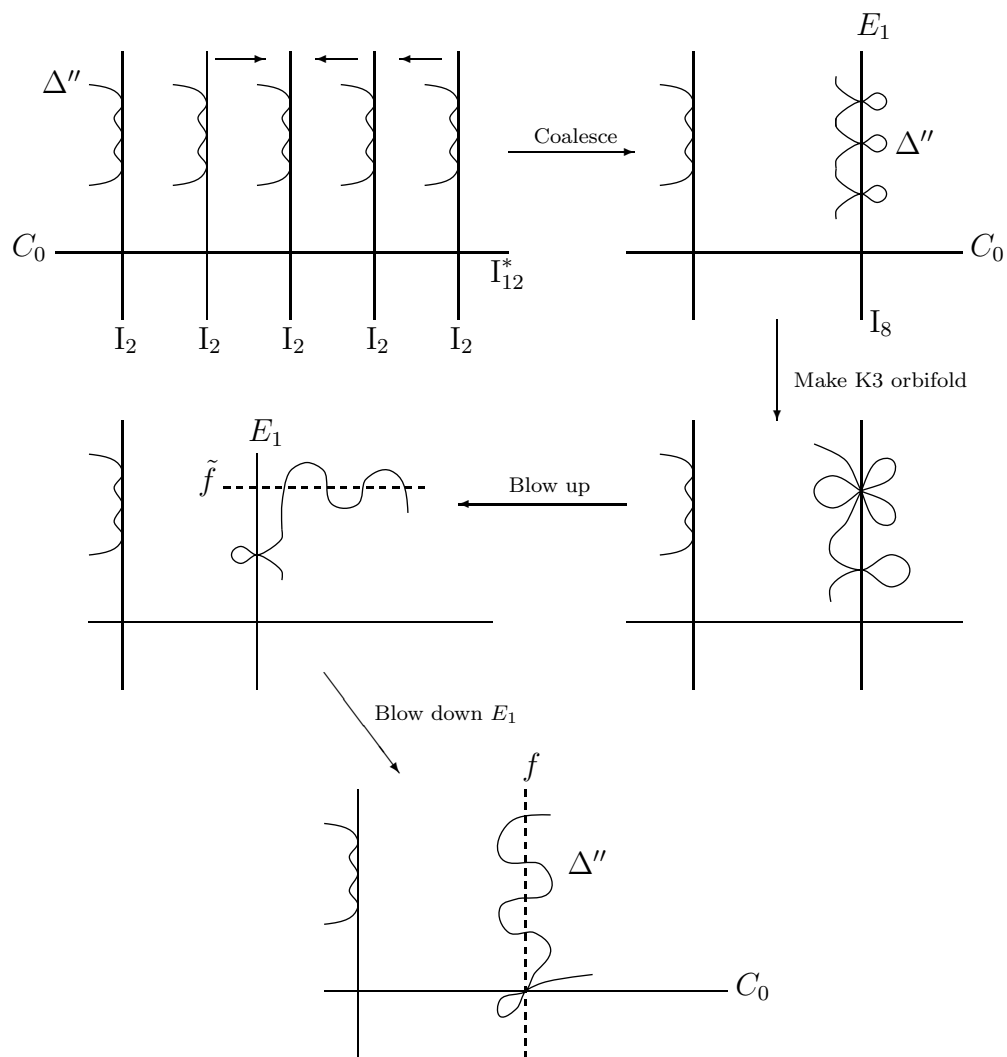


Figure 4: Four simple instantons collide with an orbifold point.

at an orbifold point. Let us consider  $k$  simple instantons. The collision of the associated  $f$ -curve with  $\Delta''$  was given in (16). Consider the zeros of  $p$  as  $s$  is varied to move along  $f$ . Generically the zeros are isolated. It is evident from the above discussion that the orbifold condition amounts to  $p$  having a zero of order two. We are therefore interested in a collision roughly of the form

$$\begin{aligned} a &= t^k - 3s^4 \\ b &= -s^2(t^k - 2s^4) \\ \delta &= t^{2k}(4t^k - 9s^4). \end{aligned} \tag{25}$$

Adding the degrees of  $s$  and  $t$  together we see that  $(a, b, \delta)$  have degrees  $(\min(k, 4), 2 + \min(k, 4), 2k + \min(k, 4))$  respectively. Thus we hit the required  $(4, 6, 12)$  for a massless tensor precisely when  $k \geq 4$ . That is, fewer than four simple instantons at an orbifold point are not enough to produce the massless tensor.

## 7 Coalesced Instantons

Now we know that four simple instantons at an orbifold point produce a massless tensor which connects the theory to a hidden obstructer, the natural question to ask is what happens when more than four simple instantons coalesce at an orbifold point. This is equivalent to asking what happens when a simple instanton hits a hidden obstructer. It is then natural to ask what happens when two hidden obstructers coalesce.

### 7.1 A simple instanton meets a hidden obstructer

Let  $k$  simple instantons hit a hidden obstructer. Recall that a hidden obstructer corresponds to a collision of  $\Delta''$  with  $C_0$ . Consider the  $f$ -curve passing through this collision point. From our discussion of simple instantons above and their relationship to  $M_1$ , it is clear that we require  $M_1$  to contain  $k$  times this  $f$ -curve. That is,  $\Delta$  includes  $2k$  times this  $f$ -curve.

Following (18), the form of the discriminant is

$$\delta = s^{18}t^{2k}(4s^6 - 9p_1^2), \tag{26}$$

where  $p_1$ , where  $s = 0$ , has a single zero at  $t = 0$ . Adding the degrees of  $s$  and  $t$  together gives the degrees of  $(a, b, \delta)$  equal to  $(4, 6, 20 + 2k)$ . Thus we have a blow-up in the base. Now the degrees along the exceptional divisor,  $E_1$ , are  $(0, 0, 8 + 2k)$ . This gives a gauge symmetry  $\mathfrak{sp}(4 + k)$ . The proper transform,  $\tilde{f}$ , of the  $f$ -curve that passed through the collision is still a line of  $I_{2k}$  fibres and so we also have an  $\mathfrak{sp}(k)$  gauge symmetry. Since this curve hits  $E_1$ , we expect a hypermultiplet in the  $(\mathbf{8} + \mathbf{2k}, \mathbf{2k})$  representation of the  $\mathfrak{sp}(4 + k) \oplus \mathfrak{sp}(k)$  part of the gauge algebra. This can be seen by applying monodromy to the results of [29]. We

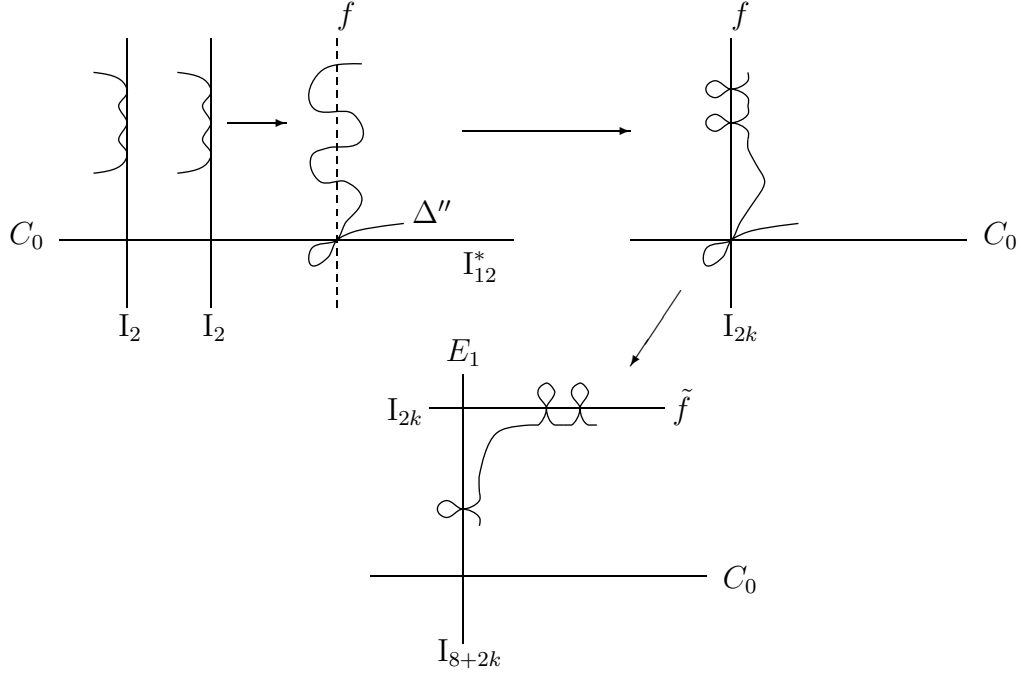


Figure 5:  $k$  simple instantons hitting a hidden obstructer.

also have hypermultiplets from the  $C_0$  collision with  $E_1$  but there is no collision between  $C_0$  and  $\tilde{f}$ .

As shown in figure 5 and the appendix, there are only two points of monodromy in the curves generating both the  $\mathfrak{sp}(4+k)$  and the  $\mathfrak{sp}(k)$  gauge algebras. Thus we have no hypermultiplets in the  $\mathbf{A}_2$  representations of either of these algebras.

As an example, suppose  $k_1$  of the simple point-like instantons collide with one of the  $4-n$  hidden obstructers and let the remaining  $8+4n-k_1$  clump into groups of  $k_i$ ,  $i = 2 \dots \mu$ . The spectrum is

1. A gauge algebra

$$\mathfrak{so}(32) \oplus \mathfrak{sp}(k_1) \oplus \mathfrak{sp}(4+k_1) \oplus \left( \bigoplus_{i=2}^{\mu} \mathfrak{sp}(k_i) \right) \oplus \mathfrak{sp}(4)^{\oplus(3-n)}. \quad (27)$$

2.  $5-n$  massless tensor supermultiplets (including the dilaton).
3. Hypermultiplets (or half-hypermultiplets if the representation is not complex) in the

following representations:

$$\begin{aligned}
(32, 8 + 2\mathbf{k}_1) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(4 + k_1) \\
(8 + 2\mathbf{k}_1, 2\mathbf{k}_1) & \text{ of } \mathfrak{sp}(4 + k_1) \oplus \mathfrak{sp}(k_1) \\
(32, 2\mathbf{k}_i) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(k_i) \\
\mathbf{k}_i(2\mathbf{k}_i - 1) - 1 & \text{ of } \mathfrak{sp}(k_i) \\
(32, 8) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(4) \quad (3 - n \text{ times}),
\end{aligned} \tag{28}$$

where  $i = 2 \dots \mu$ , as well as  $20 + (\mu - 1) - (4 - n)$  chargeless hypermultiplets.

The reader may check that anomalies cancel.

## 7.2 Two hidden obstructers meet

The natural thing to identify with two coalesced hidden obstructers is when two of the zeroes of  $p_1$  in (18) coalesce. This may be achieved by putting  $p_1 = t^2 + \alpha st + \beta s^2$  for some generic  $\alpha, \beta$ . We obtain a total degree for  $(a, b, \delta)$  at  $s = t = 0$  equal to  $(6, 9, 22)$  respectively. When we blow up this point, we obtain the exceptional divisor,  $E_1$ , with degrees  $(2, 3, 10)$ . Thus  $E_1$  is a curve of  $I_4^*$ . There is no monodromy within this and so a gauge algebra  $\mathfrak{so}(16)$  results.

We are not done however. The collision of the curve of  $I_{12}^*$  fibres along  $C_0$  and  $I_4^*$  fibres along  $E_1$  has total degree  $(4, 6, 28)$ . Therefore we are required to blow-up this point too. This introduces an exceptional divisor  $E_2$ . Along this curve, the degrees are  $(0, 0, 16)$ . In this case there is monodromy and so the gauge algebra is  $\mathfrak{sp}(8)$ . Finally there are two collision of  $E_1$  with the proper transform of  $\Delta''$  which also require blowing up. The collisions each have total degree  $(4, 6, 12)$  and so the resulting two exceptional divisors,  $E_3$  and  $E_4$ , carry smooth fibres and hence no further gauge algebra. See figure 6 for this process.

It is not much harder to go directly to the case of  $k$  more simple instantons joining the two coalesced hidden obstructers. In this case the  $f$ -curve passing through the complicated collision of  $\Delta''$  with  $C_0$  will now carry  $I_{2k}$ . Now the blow-up process is similar to the above case except that more singular fibres appear. This process is shown in figure 7.

Let us give the spectrum that results in this case. Let the remaining  $8 + 4n - k$  simple instantons that have not joined the hidden obstructers be disjoint. It is an easy matter to generalize to the case where these coalesce amongst themselves but it will clutter the notation. The result is

1. A gauge algebra

$$\mathfrak{sp}(1)^{\oplus(8+4n-k)} \oplus \mathfrak{so}(32) \oplus \mathfrak{sp}(8 + k) \oplus \mathfrak{so}(16 + 4k) \oplus \mathfrak{sp}(k)^{\oplus 3} \oplus \mathfrak{sp}(4)^{\oplus(2-n)}. \tag{29}$$

2.  $7 - n$  massless tensor supermultiplets (including the dilaton).

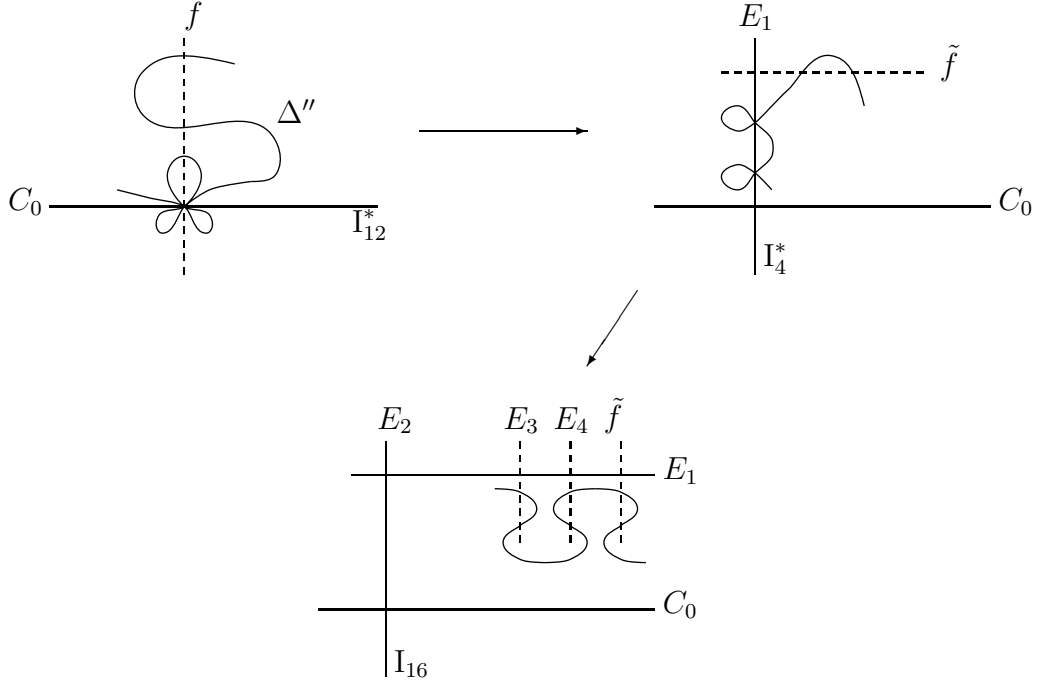


Figure 6: Two hidden obstructers at the same point.

3. Hypermultiplets (or half-hypermultiplets if the representation is not complex) in the following representations:

$$\begin{aligned}
 (32, 2) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(1) & (8 + 4n - k \text{ times}) \\
 (32, 16 + 2k) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(8 + k) \\
 (16 + 4k, 16 + 2k) & \text{ of } \mathfrak{so}(16 + 4k) \oplus \mathfrak{sp}(8 + k) & (30) \\
 (16 + 4k, 2k) & \text{ of } \mathfrak{so}(16 + 4k) \oplus \mathfrak{sp}(k) & (3 \text{ times}) \\
 (32, 8) & \text{ of } \mathfrak{so}(32) \oplus \mathfrak{sp}(4) & (2 - n \text{ times}),
 \end{aligned}$$

as well as  $22 + 5n - k$  chargeless hypermultiplets.

As usual the anomalies miraculously cancel.

A couple of points are worth noting. Firstly the gauge group is getting pretty large. For example, putting  $n = 2$  and  $k = 16$  in the above yields a rank 128 gauge group. It also contains an  $\mathfrak{so}(80)$  factor in this case. This is interesting as we know that a rank 40 gauge symmetry can never be understood perturbatively. Therefore, there is no heterotic string theory dual to our model whose conformal field theory knows about this gauge symmetry factor.

Secondly the counting of moduli, i.e., chargeless hypermultiplets is curious. This should be equal to the number of deformations of the K3, plus the number of deformations of the



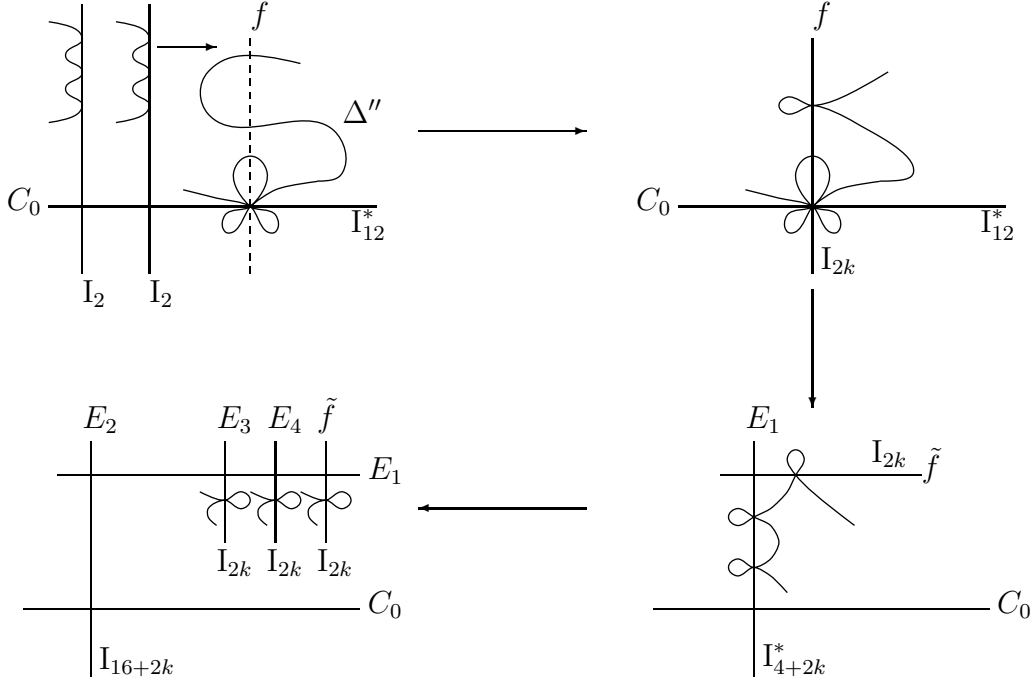


Figure 7: Two hidden obstructers meet  $k$  simple instantons.

simple instantons still free, minus the number of hidden obstructers, minus the number of moduli required to force the two obstructers to meet. The fact that there are  $22 + 5n - k$  moduli shows that this latter number of moduli, required to be tuned to make two obstructers meet, is equal to two.

This tuning must correspond to bending the K3 around as to bring two orbifold points together in the right way. This should result in a more complicated orbifold singularity. It looks as if the number of blow-ups required to smooth this orbifold singularity is equal to two, from the blow-up modes we already had, plus two more from the tuning required. This suggests that the resulting quotient singularity is either of the type  $A_4$  (i.e.,  $\mathbb{C}^2/\mathbb{Z}_5$ ) or  $D_4$  (i.e.,  $\mathbb{C}^2$  divided by the discrete quaternion group). It would be interesting to study this further.

If we continue further and attempt to bring three hidden obstructers together by giving  $p_1$  a triple zero, we obtain a total degree at the collision equal to  $(8, 12, 24)$ . After blowing up this point, the degrees along the exceptional divisor are  $(4, 6, 12)$ . While degrees greater than, or equal to,  $(4, 6, 12)$  are admissible at points within the discriminant, they are not acceptable along curves. The Calabi–Yau condition is violated if we attempt to blow up. We therefore have no further extremal transitions associated to three colliding hidden obstructers.

## Acknowledgements

I thank M. Gross for explaining to me much of the technology of elliptic fibrations used in this paper. It is also a pleasure to thank O. Aharony, C. Johnson, S. Kachru, D. Morrison, J. Polchinski, N. Seiberg and E. Silverstein for useful conversations. The author is supported by DOE grant DE-FG02-96ER40959.

## Appendix

Let  $X$  be an algebraic threefold which admits an elliptic fibration,  $p : X \rightarrow \Theta$ , for some complex surface,  $\Theta$ . The elliptic fibres degenerate over the discriminant,  $\Delta \subset \Theta$ . At a smooth point in  $\Delta$ , the bad fibres are classified by the Weierstrass classification (see, for example, [6]). In general however  $\Delta$  has singularities, usually formed by intersections of irreducible components of  $\Delta$ . In this appendix we discuss what happens to the bad fibre over such singularities in  $\Delta$ .

This problem was studied by Miranda in [30]. It has also been analyzed in [21] in terms of Tate's algorithm. We will adopt Miranda's method as it is slightly better suited to our approach and yields some Euler characteristics which are required for some points in the main text. Part of Miranda's approach was to blow up  $\Delta$  until it had only double points. In other words, he needed only to consider *transverse* collisions of two curves within  $\Delta$ . Such collisions are classified by the generic fibre type over each of the two curves. In addition some collisions could be reduced to other types by blowing up the double point. As such he needed only to consider a subset of all possible collisions.

Our problem is not quite the same as Miranda's. Blowing up the base,  $\Theta$ , will affect the canonical class of  $X$ , which we want to be trivial. Sometimes one *must* blow up the base (as in many example in the main text) in order to achieve  $K_X = 0$ . In many other cases blowing up the base would destroy  $K_X = 0$ . We find therefore that Miranda's classification is not sufficient for us. We must often deal with collisions within  $\Delta$  without blowing them up. As such there are considerably many more possibilities than Miranda considered. See [31] for a discussion of some aspects of F-theory which do fall into Miranda's classification.

Fortunately Miranda's methods did not rely on the assumption that  $\Delta$  contained only double points. Let us review the construction. Begin with the case of a complex *surface*,  $S$ , which is an elliptic fibration,  $\pi : S \rightarrow B$ , where  $B$  is an algebraic curve. Let  $z$  be an affine coordinate in  $B$ . If this fibration has a global section then we may write the fibration in Weierstrass form

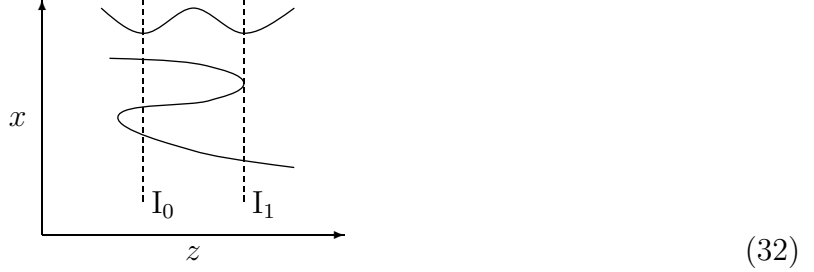
$$y^2 = x^3 + a(z)x + b(z). \tag{31}$$

The discriminant is then given by  $\delta = 4a^3 + 27b^2$ .

An elliptic curve may be written as a double cover of  $\mathbb{P}^1$  branched at four points. Indeed, the Weierstrass form exhibits this property —  $y$  has two solutions for any  $x$  except at the

roots of the right hand side of (31). There are three roots of this cubic plus one solution “at infinity”. As such  $S$  may be considered as a double cover of a  $\mathbb{P}^1$ -bundle over  $B$  branched over the curve  $x^3 + a(z)x + b(z)$  and the global section at infinity.

We may draw a typical model for  $S$  as



In this graph, the solid lines represent the branch locus (with the section at infinity at the top) and the two dotted lines represent  $\mathbb{P}^1$  fibres for fixed values of  $z$ . The generic fibre on the left intersects the branch locus 4 times. The double cover of this is a smooth elliptic. This is an  $I_0$  fibre. On the fibre on the right, two of the branch points have coalesced. This amounts to shrinking a cycle in the elliptic down to a point and, as such, is a curve with a double point. This is an  $I_1$  fibre.  $\delta$  will have a single zero at this point in  $B$ . Even though the  $I_1$  fibre is itself singular,  $S$  is smooth.

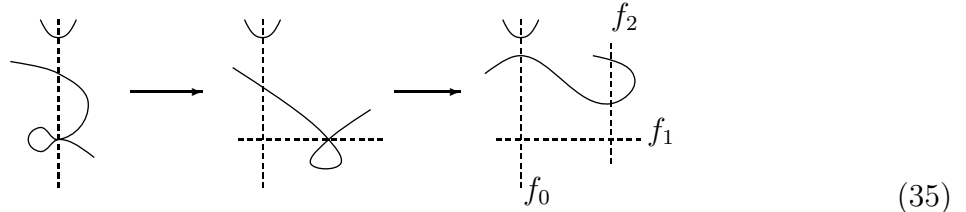
It is possible for the branch locus to degenerate further to produce higher zeros in  $\delta$ . As an example let

$$y^2 = x^3 - 3x + 2 + z^N. \quad (33)$$

If  $N$  is even this looks like



Now  $S$  is singular at  $(x, y, z) = (1, 0, 0)$ . We may resolve  $S$  by blowing this point up. We may follow the blow up in our picture. For example in the case  $N = 4$  we have



The new curves may, or may not, be in the branch locus. The rule is that they are in the branch locus if and only if the total degree of branch divisor at the point blown up is odd.

In the above case this degree is always two. We denote the fact that the new curves are not in the branch locus by drawing them as dotted lines.  $S$  will be smooth when the branch locus is smooth and so after these two blow-ups we are done. The curve  $f_0$  is the proper transform of the original bad fibre. Note that it only intersects the branch locus twice. Thus, the double cover of this is a rational curve, rather than an elliptic. The new curve  $f_2$  is also branched twice and so will map to a rational curve in the double cover.  $f_1$  is not branched at all and so must map to *two* rational curves in the double cover. The resulting configuration of curves in the double cover is

$$\begin{array}{ccccc}
 & f_0 & & f_2 & \\
 & | & & | & \\
 - & | & - & | & f_1 \\
 & | & & | & \\
 - & | & - & | & f_1 \\
 & | & & | & 
 \end{array} \tag{36}$$

This is Kodaira's  $I_4$  fibre. Subtracting the curve  $f_0$ , which was already in  $S$ , we see that the exceptional divisor within  $S$  produced by the blow-up is a chain of three  $\mathbb{P}^1$ 's. This is the resolution of the surface singularity  $A_3$  in the usual  $A$ - $D$ - $E$  classification. This method of using a double cover is probably the best for finding the blow-ups of surface singularities and may be applied to all of the  $A$ - $D$ - $E$  series. The type of bad fibres can be classified according to the degree of vanishing of  $a$ ,  $b$ , and  $\delta$ .

Now let us return to our elliptic threefold,  $X$ . Let  $s$  and  $t$  be affine coordinates in the base,  $\Theta$ . Over a generic point in the discriminant we may put  $z$  equal to a generic linear combination of  $s$  and  $t$  and reduce to the elliptic surface case. There is nothing to stop us putting such a generic slice through a bad point in the discriminant. The degrees of  $(a, b, \delta)$  will jump at such a point.

Consider a transverse intersection of two curves,  $D_1$  and  $D_2$ , within  $\Delta$ . The degrees of  $(a, b, \delta)$  in our generic slice given by  $z$  will then simply be the sum of the corresponding degrees along  $D_1$  and  $D_2$ . One may expect them to be higher for non-transverse intersections however.

For example, let us consider the case given by (16) of a curve of  $I_1$  fibres colliding with a curve of  $I_{2k}$  fibres given by

$$\begin{aligned}
 a &= t^k - 3s^2 \\
 b &= -s(t^k - 2s^2) \\
 \delta &= t^{2k}(4t^k - 9s^2).
 \end{aligned} \tag{37}$$

The degrees along  $4t^k - 9s^2 = 0$  are  $(0, 0, 1)$  (for an  $I_1$  fibre) and along  $t = 0$  are  $(0, 0, 2k)$  (for an  $I_{2k}$  fibre). At  $s = t = 0$  these curves collide and the total degrees are  $(2, 3, 2k + 2)$ , assuming  $k \geq 2$ , (which is an  $I_{2k-4}^*$  fibre).

To resolve  $X$  we certainly need to begin by blowing up the fibres along the generic parts of  $\Delta$  as in the surface case. Each time we do a blow-up of the generic points, the fibres at

the collisions will also be partially resolved. In some simple cases the fibres at the collisions will automatically be fully resolved as usual by this process but, more usually, we will only end up with a partial resolution. At this point in the resolution process,  $X$  may already be smooth or it may require a “small resolution” at the collision. Occasionally it cannot be resolved but this will not happen in any examples here.

Let us follow our example for this process. The  $I_1$  fibres require no blow-ups so we just have to consider the  $I_{2k}$  blow-up. Let us assume  $k = 2$  so that the resolution follows the sequence given in (35). In this case the partial resolution of the  $I_0^*$  fibre at the collision proceeds as

(38)

Note that the first blow-up occurs at a degree 3 point in the branch locus. The exceptional curve is therefore in the branch locus. Note that at the end the branch locus is still colliding with itself. If this were a generic point in the discriminant we would have to continue the blow-up. In this case however, we are done.  $X$  is now smooth. The fibre over the collision is the double cover of this which is given as follows:

(39)

This has Euler characteristic 4. For general  $k$  the Euler characteristic is  $2 + k$ .

We are now in a position to read off the monodromy. Note that  $f_1$  appears within the branch locus at the collision point and yet there are two  $f_1$  curves in the  $I_4$  fibre away from the collision. So long as the Weierstrass form gives  $y^2$  as a generic function of  $s$  then an orbit around  $s = 0$  within the  $t = 0$  line will exchange the two  $f_1$  curves in the fibre. Thus, this particular collision will induce monodromy. In general this collision will produce the expected monodromy in  $I_{2k}$  fibres to produce  $\mathfrak{sp}(k)$ .

One should contrast this with to a *transverse* collision of an  $I_m$ -curve and an  $I_n$ -curve in which case there is no monodromy (unless, of course, it's induced by a collision elsewhere). Analysis in [30] shows that the Euler characteristic of the fibre over such a collision is  $m + n$ . Miranda also considered the case of an  $I_{2k}$ -curve collision with a  $I_{2m}^*$ -curve which is relevant for our purposes. In this case the resulting fibre at the collision point has Euler characteristic  $2m + k + 6$ . Monodromy is induced on the  $I_{2k}$  fibre but not the  $I_{2m}^*$  fibre.

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